Ponavljanje za pismeni ispit – trigonometrijski identiteti

1. Koristeći osnovne trigonometrijske relacije, pojednostavni izraz $\frac{1}{(1-\text{tg}x)^2+(1+\text{tg}x)^2}$.

$$\frac{1}{(1-\text{tg}x)^2+(1+\text{tg}x)^2}=\frac{1}{1-2\text{tg}x+\text{tg}^2x+1+2\text{tg}x+\text{tg}^2x}=\frac{1}{2+2\text{tg}^2x}=\frac{1}{2\left(1+\text{tg}^2x\right)}=$$

$$\frac{1}{2\left(1+\frac{\sin^2x}{\cos^2x}\right)}=\frac{1}{2\left(\frac{\cos^2x+\sin^2x}{\cos^2x}\right)}=\frac{1}{2\left(\frac{1}{\cos^2x}\right)}=\frac{\frac{1}{1}}{\frac{2}{\cos^2x}}=\boxed{\frac{\cos^2x}{2}}$$

2. Koristeći osnovne trigonometrijske relacije, dokaži da vrijedi jednakost $\frac{\cos t}{1-\cos t}-\frac{\cos t}{1+\cos t}=2\text{ctg}^2t$.

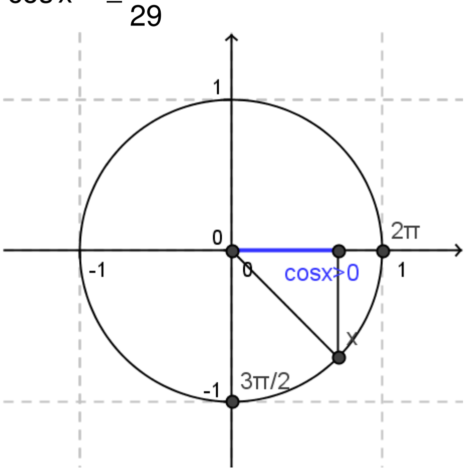
$$\boxed{\frac{\cos t}{1-\cos t}-\frac{\cos t}{1+\cos t}}=\frac{\cos t(1+\cos t)-\cos t(1-\cos t)}{(1-\cos t)(1+\cos t)}=\frac{\cos t+\cos^2t-\cos t+\cos^2t}{1-\cos^2t}=$$

$$\frac{2\cos^2t}{\sin^2t}=\boxed{2\text{ctg}^2t}$$

3. Ako je $\sin x=-\frac{21}{29}$, $x\in\left\langle\frac{3\pi}{2},2\pi\right\rangle$ i $\text{tg}y=\sqrt{3}$, $y\in\left\langle\pi,\frac{3\pi}{2}\right\rangle$, odredi:

   a. $\cos x$,

$$\sin^2x+\cos^2x=1$$
$$\left(-\frac{21}{29}\right)^2+\cos^2x=1$$
$$\frac{441}{841}+\cos^2x=1$$
$$\cos^2x=1-\frac{441}{841}$$
$$\cos^2x=\frac{841-441}{841}$$
$$\cos^2x=\frac{400}{841}\quad /\sqrt{\ }$$
$$\cos x=\pm\frac{20}{29}$$



$$\boxed{\cos x = \frac{20}{29}}$$

b. tgx ,

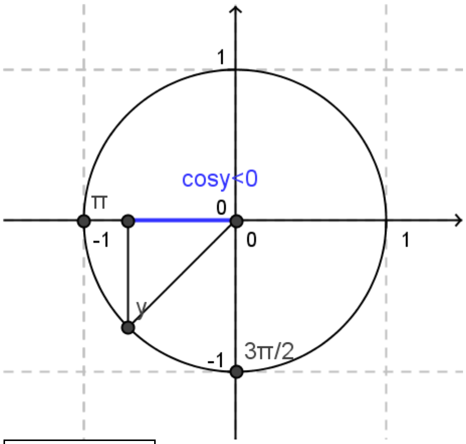
$$tgx = \frac{\sin x}{\cos x}$$
$$tgx = \frac{-\frac{21}{29}}{\frac{20}{29}}$$
$$tgx = \frac{-\frac{21}{1}}{\frac{20}{1}}$$
$$\boxed{tgx = -\frac{21}{20}}$$

c. $ctgx$,

$$ctgx = \frac{1}{tgx}$$
$$\boxed{ctgx = -\frac{20}{21}}$$

d. $\cos y$,

$$\sin^2 y + \cos^2 y = 1 \quad / : \cos^2 y$$
$$tg^2 y + 1 = \frac{1}{\cos^2 y}$$
$$\left(\sqrt{3}\right)^2 + 1 = \frac{1}{\cos^2 y}$$
$$3 + 1 = \frac{1}{\cos^2 y}$$
$$4 = \frac{1}{\cos^2 y}$$
$$\cos^2 y = \frac{1}{4} \quad / \sqrt{\ }$$
$$\cos y = \pm\frac{1}{2}$$



$$\boxed{\cos y=-\frac{1}{2}}$$

e. $\sin y$,

$$tgy=\frac{\sin y}{\cos y}$$
$$\sin y=\cos y\cdot tgy$$
$$\sin y=-\frac{1}{2}\cdot\sqrt{3}$$
$$\boxed{\sin y=-\frac{\sqrt{3}}{2}}$$

f. $ctgy$,

$$ctgy=\frac{1}{tgy}$$
$$ctgy=\frac{1}{\sqrt{3}}$$
$$ctgy=\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$
$$\boxed{ctgy=\frac{\sqrt{3}}{3}}$$

g. $\sin(x-y)$,

$$\sin(x-y)=\sin x\cdot\cos y-\cos x\cdot\sin y=-\frac{21}{29}\cdot\left(-\frac{1}{2}\right)-\frac{20}{29}\cdot\left(-\frac{\sqrt{3}}{2}\right)=\frac{21}{58}+\frac{20\sqrt{3}}{58}=$$
$$\boxed{\frac{21+20\sqrt{3}}{58}}$$

h. $\cos\left(x+\frac{\pi}{6}\right)$,

$$\cos\left(x+\frac{\pi}{6}\right)=\cos x\cdot\cos\frac{\pi}{6}-\sin x\cdot\sin\frac{\pi}{6}=\frac{20}{29}\cdot\frac{\sqrt{3}}{2}+\frac{21}{29}\cdot\frac{1}{2}=\frac{20\sqrt{3}}{58}+\frac{21}{58}=$$
$$\boxed{\frac{20\sqrt{3}+21}{58}}$$

i. $\mathrm{tg}(x+y)$,

$$\mathrm{tg}(x+y)=\frac{\mathrm{tg}x+\mathrm{tg}y}{1-\mathrm{tg}x\cdot\mathrm{tg}y}=\frac{-\frac{21}{20}+\sqrt{3}}{1+\frac{21}{20}\cdot\sqrt{3}}=\frac{-\frac{21}{20}+\frac{\sqrt{3}}{1}}{\frac{1}{1}+\frac{21\sqrt{3}}{20}}=\frac{\frac{-21+20\sqrt{3}}{20}}{\frac{20+21\sqrt{3}}{20}}=\frac{\frac{-21+20\sqrt{3}}{1}}{\frac{20+21\sqrt{3}}{1}}=$$

$$\frac{-21+20\sqrt{3}}{20+21\sqrt{3}}=\frac{-21+20\sqrt{3}}{20+21\sqrt{3}}\cdot\frac{20-21\sqrt{3}}{20-21\sqrt{3}}=\frac{-420+441\sqrt{3}+400\sqrt{3}-420\left(\sqrt{3}\right)^2}{20^2-\left(21\sqrt{3}\right)^2}=$$

$$\frac{-420+841\sqrt{3}-420\cdot 3}{400-441\cdot 3}=\frac{-420+841\sqrt{3}-1260}{400-1323}=\frac{-1680+841\sqrt{3}}{-923}=$$

$$\boxed{\frac{1680-841\sqrt{3}}{923}}$$

j. $\mathrm{ctg}\left(\frac{\pi}{4}-y\right)$,

$$\mathrm{ctg}\left(\frac{\pi}{4}-y\right)=\frac{\mathrm{ctg}\frac{\pi}{4}\cdot\mathrm{ctg}y+1}{\mathrm{ctg}y-\mathrm{ctg}\frac{\pi}{4}}=\frac{1\cdot\frac{\sqrt{3}}{3}+1}{\frac{\sqrt{3}}{3}-1}=\frac{\frac{\sqrt{3}}{3}+\frac{1}{1}}{\frac{\sqrt{3}}{3}-\frac{1}{1}}=\frac{\frac{\sqrt{3}+3}{3}}{\frac{\sqrt{3}-3}{3}}=\frac{\frac{\sqrt{3}+3}{1}}{\frac{\sqrt{3}-3}{1}}=\frac{\sqrt{3}+3}{\sqrt{3}-3}=$$

$$\frac{\sqrt{3}+3}{\sqrt{3}-3}\cdot\frac{\sqrt{3}+3}{\sqrt{3}+3}=\frac{\left(\sqrt{3}+3\right)^2}{\left(\sqrt{3}\right)^2-3^2}=\frac{\left(\sqrt{3}\right)^2+2\cdot\sqrt{3}\cdot 3+3^2}{3-9}=\frac{3+6\sqrt{3}+9}{-6}=\frac{12+6\sqrt{3}}{-6}=$$

$$\frac{12}{-6}+\frac{6\sqrt{3}}{-6}=\boxed{-2-\sqrt{3}}$$

k. $\sin(2x)$,

$$\sin(2x)=2\sin x\cos x=\frac{2}{1}\left(-\frac{21}{29}\right)\cdot\frac{20}{29}=\boxed{-\frac{840}{841}}$$

l. $\cos(2y)$,

$$\cos(2y)=\cos^2 y-\sin^2 y=\left(\frac{20}{29}\right)^2-\left(-\frac{21}{29}\right)^2=\frac{400}{841}-\frac{441}{841}=\frac{400-441}{841}=\boxed{-\frac{41}{841}}$$

m. $\mathrm{tg}(2x)$,

$$\mathrm{tg}(2x)=\frac{2\mathrm{tg}x}{1-\mathrm{tg}^2x}=\frac{2\left(-\frac{21}{20}\right)}{1-\left(-\frac{21}{20}\right)^2}=\frac{-\frac{42}{20}}{\frac{1}{1}-\frac{441}{400}}=\frac{-\frac{42}{20}}{\frac{400-441}{400}}=\frac{-\frac{42}{20}}{\frac{-41}{400}}=\frac{-\frac{42}{1}}{\frac{-41}{20}}=\boxed{\frac{840}{41}}$$

n. $\mathrm{ctg}(2y)$,

$$\operatorname{ctg}(2y)=\frac{\operatorname{ctg}^2 y-1}{2\operatorname{ctg} y}=\frac{\left(\frac{\sqrt{3}}{3}\right)^2-1}{2\cdot\frac{\sqrt{3}}{3}}=\frac{\frac{3}{9}-1}{\frac{2\sqrt{3}}{3}}=\frac{\frac{1}{3}-\frac{1}{1}}{\frac{2\sqrt{3}}{3}}=\frac{\frac{1-3}{3}}{\frac{2\sqrt{3}}{3}}=\frac{\frac{-2}{3}}{\frac{2\sqrt{3}}{3}}=\frac{\frac{-1}{1}}{\frac{\sqrt{3}}{1}}=-\frac{1}{\sqrt{3}}=$$

$$-\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\boxed{-\frac{\sqrt{3}}{3}}$$

o. $\sin\left(\frac{y}{2}\right)$,

$$\sin^2\left(\frac{y}{2}\right)=\frac{1-\cos y}{2}$$

$$\sin^2\left(\frac{y}{2}\right)=\frac{1-\left(-\frac{1}{2}\right)}{2}$$

$$\sin^2\left(\frac{y}{2}\right)=\frac{1+\frac{1}{2}}{2}$$

$$\sin^2\left(\frac{y}{2}\right)=\frac{\frac{2+1}{2}}{2}$$

$$\sin^2\left(\frac{y}{2}\right)=\frac{\frac{3}{2}}{\frac{2}{1}}$$

$$\sin^2\left(\frac{y}{2}\right)=\frac{3}{4}\quad /\sqrt{}$$

$$\sin\left(\frac{y}{2}\right)=\pm\frac{\sqrt{3}}{2}$$

$$y\in\left\langle\pi,\frac{3\pi}{2}\right\rangle\Rightarrow\frac{y}{2}\in\left\langle\frac{\pi}{2},\frac{3\pi}{4}\right\rangle\Rightarrow\sin\left(\frac{y}{2}\right)>0$$

$$\boxed{\sin\left(\frac{y}{2}\right)=\frac{\sqrt{3}}{2}}$$

p. $\cos\left(\frac{x}{2}\right)$,

$$\cos^2\left(\frac{x}{2}\right)=\frac{1+\cos x}{2}$$

$$\cos^2\left(\frac{x}{2}\right)=\frac{1+\frac{20}{29}}{2}$$

$$\cos^2\left(\frac{x}{2}\right)=\frac{\frac{29+20}{29}}{2}$$

$$\cos^2\left(\frac{x}{2}\right)=\frac{\frac{49}{29}}{\frac{2}{1}}$$

$$\cos^2\left(\frac{x}{2}\right)=\frac{49}{58} \quad /\sqrt{\ }$$

$$\cos\left(\frac{x}{2}\right)=\pm\frac{7}{\sqrt{58}}=\pm\frac{7}{\sqrt{58}}\cdot\frac{\sqrt{58}}{\sqrt{58}}=\pm\frac{7\sqrt{58}}{58}$$

$$x\in\left\langle\frac{3\pi}{2},2\pi\right\rangle\Rightarrow\frac{x}{2}\in\left\langle\frac{3\pi}{4},\pi\right\rangle\Rightarrow\cos\left(\frac{x}{2}\right)<0$$

$$\boxed{\cos\left(\frac{x}{2}\right)=-\frac{7\sqrt{58}}{58}}$$

q. $\mathrm{tg}\left(\frac{y}{2}\right)$,

$$\mathrm{tg}^2\left(\frac{y}{2}\right)=\frac{1-\cos y}{1+\cos y}$$

$$\mathrm{tg}^2\left(\frac{y}{2}\right)=\frac{1-\left(-\frac{1}{2}\right)}{1-\frac{1}{2}}$$

$$\mathrm{tg}^2\left(\frac{y}{2}\right)=\frac{1+\frac{1}{2}}{1-\frac{1}{2}}$$

$$\mathrm{tg}^2\left(\frac{y}{2}\right)=\frac{\frac{2+1}{2}}{\frac{2-1}{2}}$$

$$\mathrm{tg}^2\left(\frac{y}{2}\right)=\frac{\frac{3}{2}}{\frac{1}{2}}$$

$$\mathrm{tg}^2\left(\frac{y}{2}\right)=3 \quad /\sqrt{\ }$$

$$\mathrm{tg}\left(\frac{y}{2}\right)=\pm\sqrt{3}$$

$$y\in\left\langle\pi,\frac{3\pi}{2}\right\rangle\Rightarrow\frac{y}{2}\in\left\langle\frac{\pi}{2},\frac{3\pi}{4}\right\rangle\Rightarrow\mathrm{tg}\left(\frac{y}{2}\right)<0$$

$$\boxed{\mathrm{tg}\left(\frac{y}{2}\right)=-\sqrt{3}}$$

r. $\mathrm{ctg}\left(\frac{x}{2}\right)$.

$$\mathrm{ctg}^2\left(\frac{x}{2}\right)=\frac{1+\cos x}{1-\cos x}$$

$$\text{ctg}^2\left(\frac{x}{2}\right)=\frac{1+\frac{20}{29}}{1-\frac{20}{29}}$$

$$\text{ctg}^2\left(\frac{x}{2}\right)=\frac{\frac{29+20}{29}}{\frac{29-20}{29}}$$

$$\text{ctg}^2\left(\frac{x}{2}\right)=\frac{\frac{49}{29}}{\frac{9}{29}}$$

$$\text{ctg}^2\left(\frac{x}{2}\right)=\frac{49}{9} \quad /\sqrt{\ }$$

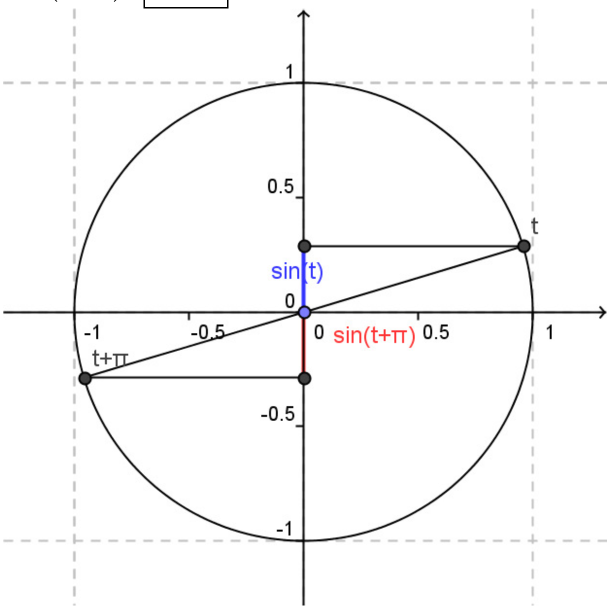
$$\text{ctg}\left(\frac{x}{2}\right)=\pm\frac{7}{3}$$

$$x\in\left\langle\frac{3\pi}{2},2\pi\right\rangle\Rightarrow\frac{x}{2}\in\left\langle\frac{3\pi}{4},\pi\right\rangle\Rightarrow\text{ctg}\left(\frac{x}{2}\right)<0$$

$$\boxed{\text{ctg}\left(\frac{x}{2}\right)=-\frac{7}{3}}$$

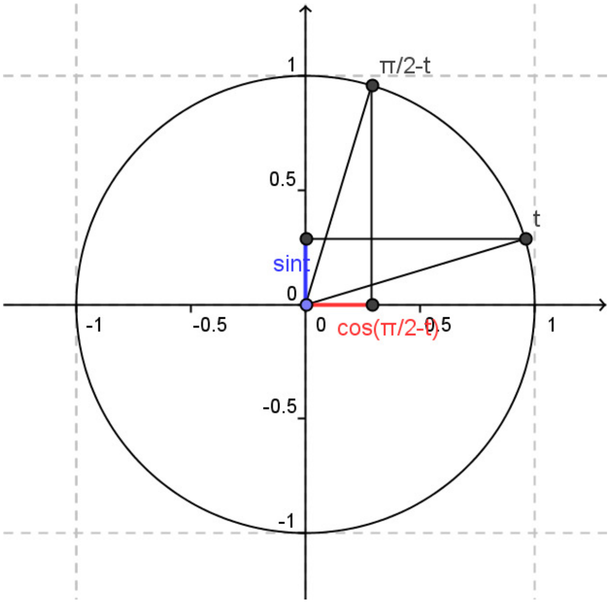
4. Pojednostavni izraze:
   a. $\sin(t+\pi)$,

$$\sin(t+\pi)=\boxed{-\sin t}$$



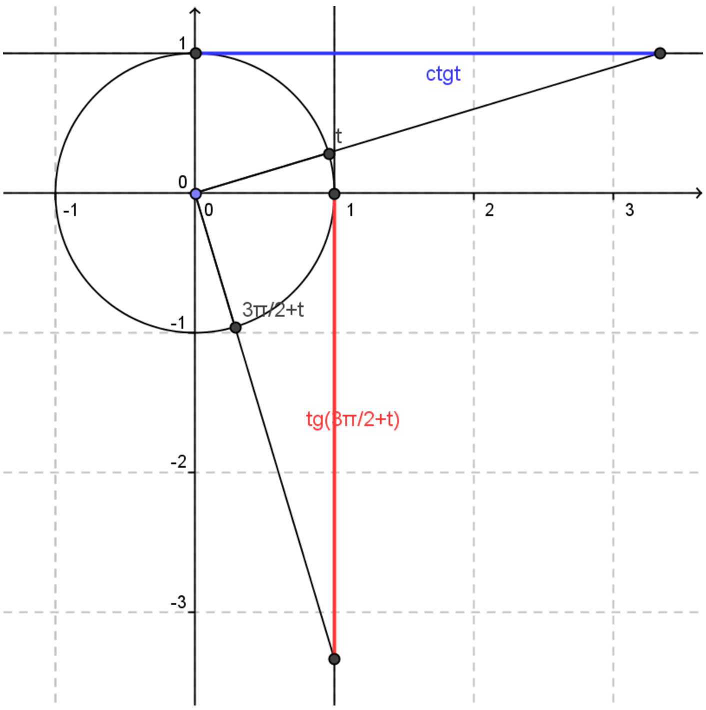
   b. $\cos\left(\frac{\pi}{2}-t\right)$,

$$\cos\left(\frac{\pi}{2}-t\right)=\boxed{\sin t}$$



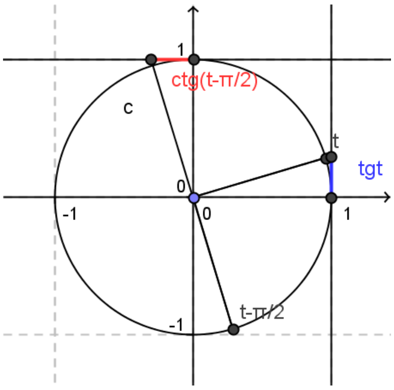
c. $\operatorname{tg}\left(\frac{3\pi}{2}+t\right)$,

$$\operatorname{tg}\left(\frac{3\pi}{2}+t\right)=\boxed{-\operatorname{ctg}t}$$



d. $\operatorname{ctg}\left(t-\frac{\pi}{2}\right)$.

$$\operatorname{ctg}\left(t-\frac{\pi}{2}\right)=\boxed{-\operatorname{tg}t}$$



5. Koristeći formule za pretvorbu umnoška trigonometrijskih funkcija u zbroj, pojednostavni izraz $\sin\left(\frac{\pi}{6}+2x\right)\cdot\sin\left(\frac{\pi}{6}-2x\right)$.

$$\sin\left(\frac{\pi}{6}+2x\right)\cdot\sin\left(\frac{\pi}{6}-2x\right)=\frac{1}{2}\left[\cos\left(\left(\frac{\pi}{6}+2x\right)-\left(\frac{\pi}{6}-2x\right)\right)-\cos\left(\left(\frac{\pi}{6}+2x\right)+\left(\frac{\pi}{6}-2x\right)\right)\right]=$$

$$\frac{1}{2}\left[\cos\left(\frac{\pi}{6}+2x-\frac{\pi}{6}+2x\right)-\cos\left(\frac{\pi}{6}+2x+\frac{\pi}{6}-2x\right)\right]=\frac{1}{2}\left[\cos(4x)-\cos\left(\frac{2\pi}{6}\right)\right]=$$

$$\frac{1}{2}\left[\cos(4x)-\cos\left(\frac{\pi}{3}\right)\right]=\frac{1}{2}\left[\cos(4x)-\frac{1}{2}\right]=\boxed{\frac{1}{2}\cos(4x)-\frac{1}{4}}$$

6. Koristeći formule za pretvorbu zbroja trigonometrijskih funkcija u umnožak, pojednostavni izraz $\frac{\cos(3x-y)-\cos(3y-x)}{\cos(2x)-\cos(2y)}$.

$$\frac{\cos(3x-y)-\cos(3y-x)}{\cos(2x)-\cos(2y)}=\frac{-2\sin\frac{(3x-y)+(3y-x)}{2}\sin\frac{(3x-y)-(3y-x)}{2}}{-2\sin\frac{2x+2y}{2}\sin\frac{2x-2y}{2}}=$$

$$\frac{\sin\frac{3x-y+3y-x}{2}\sin\frac{3x-y-3y+x}{2}}{\sin\frac{2(x+y)}{2}\sin\frac{2(x-y)}{2}}=\frac{\sin\frac{2x+2y}{2}\sin\frac{4x-4y}{2}}{\sin(x+y)\sin(x-y)}=\frac{\sin\frac{2(x+y)}{2}\sin\frac{4(x-y)}{2}}{\sin(x+y)\sin(x-y)}$$

$$=\frac{\sin(x+y)\sin[2(x-y)]}{\sin(x+y)\sin(x-y)}=\frac{\sin[2(x-y)]}{\sin(x-y)}=\frac{2\sin(x-y)\cos(x-y)}{\sin(x-y)}=\boxed{2\cos(x-y)}$$