KRUŽNICA

1) Kako glasi jednadžba kružnice kojoj je točka $S(0,-1)$ središte, a prolazi točkom $A(1,2)$?

$$S(0,-1), \quad A(1,2)$$
$$p\ q \quad\ \ x\ y$$

$$(x-p)^2 + (y-q)^2 = r^2$$
$$(1-0)^2 + (2+1)^2 = r^2$$
$$1^2 + 3^2 = r^2$$
$$1 + 9 = r^2$$
$$r^2 = 10$$
$$(x-0)^2 + (y+1)^2 = 10$$
$$\boxed{x^2 + (y+1)^2 = 10}$$

2) Napiši jednadžbu kružnice koja prolazi točkama $A(3,4)$, $B(0,5)$, $C(4,-3)$.
$$x\ yx\ yx\ y$$

$$(x-p)^2 + (y-q)^2 = r^2$$
$$(3-p)^2 + (4-q)^2 = r^2$$
$$(0-p)^2 + (5-q)^2 = r^2$$
$$\underline{(4-p)^2 + (-3-q)^2 = r^2}$$
$$\qquad\qquad \longrightarrow \quad (-3-q)^2 = (3+q)^2$$
$$\qquad\qquad\qquad\qquad\qquad = 9 + 6q + q^2$$

$$9 - 6p + p^2 + 16 - 8q + q^2 = r^2$$
$$p^2 + 25 - 10q + q^2 = r^2 \longleftarrow$$
$$16 - 8p + p^2 + 9 + 6q + q^2 = r^2$$

$$p^2 - 6p + q^2 - 8q + 25 = r^2$$
$$p^2 + q^2 - 10q + 25 = r^2 \ /\cdot(-1)$$
$$p^2 - 8p + q^2 + 6q + 25 = r^2$$

$$p^2 - 6p + q^2 - 8q + 25 = r^2 \quad |+$$
$$-p^2 - q^2 + 10q - 25 = -r^2 \quad |+$$
$$p^2 - 8p + q^2 + 6q + 25 = r^2$$

$$-6p + 2q = 0 \ /:2 \longleftarrow$$
$$-8p + 16q = 0 \ /:8 \longleftarrow$$

$$-3p + q = 0 \ /\cdot(-2)$$
$$-p + 2q = 0$$

$$6p - 2q = 0 \quad |+$$
$$-p + 2q = 0 \longleftarrow$$

$$5p = 0 \ /:5$$
$$\boxed{p = 0}$$

$$0 + 2q = 0$$
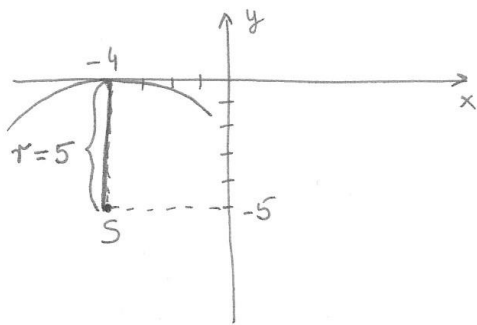$$2q = 0 \ /:2$$
$$\boxed{q = 0}$$

$$p^2 + 25 - 10q + q^2 = r^2$$
$$25 = r^2$$
$$\boxed{r^2 = 25}$$

$$(x-p)^2 + (y-q)^2 = r^2$$
$$\boxed{x^2 + y^2 = 25}$$

3) Kružnica sa središtem u $S(-4,-5)$ dira x-os. Napiši jednadžbu!

$$(x-p)^2 + (y-q)^2 = r^2$$
$$(x+4)^2 + (y+5)^2 = 5^2$$
$$\boxed{(x+4)^2 + (y+5)^2 = 25}$$

4) Odredi jednadžbu kružnice koja prolazi točkama $A(4,4)$ i $B(-3,3)$, a središte joj je na x-osi.

$$S(p,0)$$
$$(x-p)^2 + (y-q)^2 = r^2$$
$$(4-p)^2 + (4-0)^2 = r^2$$
$$(-3-p)^2 + (3-0)^2 = r^2$$
$$\underline{}$$
$$16 - 8p + p^2 + 16 = r^2$$
$$9 + 6p + p^2 + 9 = r^2$$
$$\underline{}$$
$$p^2 - 8p + 32 = r^2$$
$$p^2 + 6p + 18 = r^2 \, / \cdot (-1)$$
$$\underline{}$$
$$p^2 - 8p + 32 = r^2$$
$$-p^2 - 6p - 18 = -r^2 \, \Big| +$$
$$\underline{}$$

$$-14p + 14 = 0$$
$$-14p = -14 \, / : (-14)$$
$$\boxed{p = 1}$$

$$1 - 8 + 32 = r^2$$
$$25 = r^2$$
$$\boxed{r^2 = 25}$$

$$(x-p)^2 + (y-q)^2 = r^2$$
$$\boxed{(x-1)^2 + y^2 = 25}$$

5) Odredi jednadžbu kružnice polumjera $r=9$, koncentrične kružnici $x^2 + y^2 + 2x + 2y - 14 = 0$.

$$x^2 + y^2 + 2x + 2y - 14 = 0$$
$$x^2 + 2x + y^2 + 2y - 14 = 0$$
$$\underline{x^2 + 2x + 1} - 1 + \underline{y^2 + 2y + 1} - 1 - 14 = 0$$
$$(x+1)^2 + (y+1)^2 - 16 = 0$$
$$(x+1)^2 + (y+1)^2 = 16$$
$$S(-1,-1)$$

$$S(-1,-1), \; r = 9$$
$$(x-p)^2 + (y-q)^2 = r^2$$
$$\boxed{(x+1)^2 + (y+1)^2 = 81}$$

6) Odredi međusobni položaj pravca $y = -x + 11$ i kružnice
$(x-1)^2 + y^2 = 16$.

$$(x-1)^2 + y^2 = 16$$
$$y = -x + 11$$

$$(-x+11)^2 = (x-11)^2 = x^2 - 22x + 121$$

$$(x-1)^2 + (-x+11)^2 = 16$$
$$x^2 - 2x + 1 + x^2 - 22x + 121 = 16$$
$$x^2 - 2x + 1 + x^2 - 22x + 121 - 16 = 0$$
$$2x^2 - 24x + 106 = 0 \, / :2$$
$$x^2 - 12x + 53 = 0$$

$$D = b^2 - 4ac$$
$$D = (-12)^2 - 4 \cdot 1 \cdot 53$$
$$D = 144 - 212$$
$$D = -68 < 0$$

kvadratna jednadžba nema
realnih rješenja $\Rightarrow$

$$\boxed{\text{pravac i kružnica nemaju zajedničkih točaka}}$$

7) Odredi duljinu tetive kružnice $x^2 + y^2 = 16$ određene pravcem
$y = -x$.

$$x^2 + y^2 = 16$$
$$y = -x$$

$$x^2 + (-x)^2 = 16$$
$$x^2 + x^2 = 16$$
$$2x^2 = 16 \, /:2$$
$$x^2 = 8 \, /\sqrt{}$$
$$x = \pm\sqrt{8} = \pm 2\sqrt{2}$$

$$x_1 = 2\sqrt{2} \qquad x_2 = -2\sqrt{2}$$
$$y_1 = -2\sqrt{2} \qquad y_2 = 2\sqrt{2}$$

$$S_1(2\sqrt{2}, -2\sqrt{2}) \, , \, S_2(-2\sqrt{2}, 2\sqrt{2})$$

$$t = |S_1 S_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2\sqrt{2} - 2\sqrt{2})^2 + (2\sqrt{2} + 2\sqrt{2})^2} =$$
$$= \sqrt{(-4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{16 \cdot 2 + 16 \cdot 2} = \sqrt{32 + 32} = \sqrt{64} = 8$$

$$\boxed{t = 8}$$

8) Napiši jednadžbu tangente i normale na kružnicu $(x+1)^2 + (y-3)^2 = 169$ u njenoj točki $D(x > -1, -9)$.

$(x+1)^2 + (y-3)^2 = 169$

$(x+1)^2 + (-9-3)^2 = 169$

$(x+1)^2 + (-12)^2 = 169$

$(x+1)^2 = 169 - 144$

$(x+1)^2 = 25 \ /\sqrt{}$

$x+1 = \pm 5$

$x_1 = 5 - 1 \qquad x_2 = -5 - 1$

$x_1 = 4 \qquad \cancel{x_2 = -6}$

$D(4, -9)$

$(x_0 - p)(x - p) + (y_0 - q)(y - q) = r^2$

$(4+1)(x+1) + (-9-3)(y-3) = 169$

$5(x+1) - 12(y-3) = 169$

$5x + 5 - 12y + 36 = 169$

$-12y = -5x + 169 - 5 - 36$

$-12y = -5x + 128 \ /:(-12)$

$y = \frac{5}{12}x - \frac{128}{12}$

$t \dots \boxed{y = \frac{5}{12}x - \frac{32}{3}}$

$k_t = \frac{5}{12} \ , \quad k_n = -\frac{12}{5} \ , \quad D(4, -9)$

$y - y_1 = k(x - x_1)$

$y + 9 = -\frac{12}{5}(x - 4)$

$y + 9 = -\frac{12}{5}x + \frac{48}{5}$

$y = -\frac{12}{5}x + \frac{48}{5} - 9$

$y = -\frac{12}{5}x + \frac{48-45}{5}$

$n \dots \boxed{y = -\frac{12}{5}x + \frac{3}{5}}$

9) Odredi jednadžbe tangenti kružnice $(x-2)^2 + (y-2)^2 = 25$ paralelnih pravcu $3x - 4y - 1 = 0$.

$3x - 4y - 1 = 0$

$-4y = -3x + 1 \ /:(-4)$

$y = \frac{3}{4}x - \frac{1}{4}$

$k = \frac{3}{4}$

$(x-2)^2 + (y-2)^2 = 25$

$p = 2$

$q = 2$

$r^2 = 25$

$r^2(k^2 + 1) = (kp - q + l)^2$

$25\left(\frac{9}{16} + 1\right) = \left(\frac{3}{4} \cdot \cancel{2}^{1} - 2 + l\right)^2$

$25 \cdot \frac{9+16}{16} = \left(\frac{3}{2} - 2 + l\right)^2$

$\frac{25}{1} \cdot \frac{25}{16} = \left(\frac{3-4}{2} + l\right)^2$

$\frac{625}{16} = \left(l - \frac{1}{2}\right)^2$

$\left(l - \frac{1}{2}\right)^2 = \frac{625}{16} \ /\sqrt{}$

$l - \frac{1}{2} = \pm \frac{25}{4}$

$l - \frac{1}{2} = \frac{25}{4} \qquad\qquad l - \frac{1}{2} = -\frac{25}{4}$

$l = \frac{25}{4} + \frac{1}{2} \qquad\qquad l = -\frac{25}{4} + \frac{1}{2}$

$l = \frac{25+2}{4} \qquad\qquad l = \frac{-25+2}{4}$

$l_1 = \frac{27}{4} \qquad\qquad l_2 = -\frac{23}{4}$

$t_1 \dots \boxed{y = \frac{3}{4}x + \frac{27}{4}} \qquad t_2 \dots \boxed{y = \frac{3}{4}x - \frac{23}{4}}$

10) Napiši jednadžbe tangenti kružnice $x^2+y^2-14y+32=0$, povučenih iz točke $T(5,4)$.

$x^2+y^2-14y+32=0$

$x^2+y^2-14y+49-49+32=0$

$x^2+(y-7)^2-17=0$

$x^2+(y-7)^2=17$

$p=0, \; q=7, \; r^2=17$

$y=kx+\ell$

$T(5,4)$

$4=k\cdot5+\ell$

$4=5k+\ell$

$5k+\ell=4$

$\ell=4-5k$

$$r^2(k^2+1)=(kp-q+\ell)^2$$

$$17(k^2+1)=(k\cdot0-7+4-5k)^2$$

$$17k^2+17=(-5k-3)^2$$

$$17k^2+17=(5k+3)^2$$

$$17k^2+17=25k^2+30k+9$$

$$17k^2+17-25k^2-30k-9=0$$

$$-8k^2-30k+8=0 \quad /:(-2)$$

$$4k^2+15k-4=0$$

$$k_{1,2}=\frac{-15\pm\sqrt{225+64}}{8}=\frac{-15\pm\sqrt{289}}{8}=\frac{-15\pm17}{8}$$

$k_1=\dfrac{-15+17}{8}=\dfrac{2}{8}=\dfrac{1}{4} \quad , \qquad k_2=\dfrac{-15-17}{8}=\dfrac{-32}{8}=-4$

$\ell_1=4-5\cdot\dfrac{1}{4}$

$\quad =4-\dfrac{5}{4}$

$\quad =\dfrac{16-5}{4}$

$\quad =\dfrac{11}{4}$

$\ell_2=4-5\cdot(-4)$

$\quad =4+20$

$\quad =24$

$t_1\ldots \boxed{y=\dfrac{1}{4}x+\dfrac{11}{4}}$ $\qquad\qquad$ $t_2\ldots \boxed{y=-4x+24}$

ELIPSA

1) Odredi kanonsku jednadžbu, veliku i malu poluos, linearni
   ekscentricitet, tjemena i žarišta te skiciraj elipsu danu jednadžbom
   $x^2 + 4y^2 = 25$.

$x^2 + 4y^2 = 25 \;/:25$

$\dfrac{x^2}{25} + \dfrac{4y^2 \;/:4}{25 \;/:4} = 1$

$\dfrac{x^2}{25} + \dfrac{y^2}{\frac{25}{4}} = 1$

$a^2 = 25 \Rightarrow \boxed{a=5}$

$b^2 = \dfrac{25}{4} \Rightarrow \boxed{b = \dfrac{5}{2}}$

$e^2 = a^2 - b^2$

$e^2 = 5^2 - \left(\dfrac{5}{2}\right)^2$
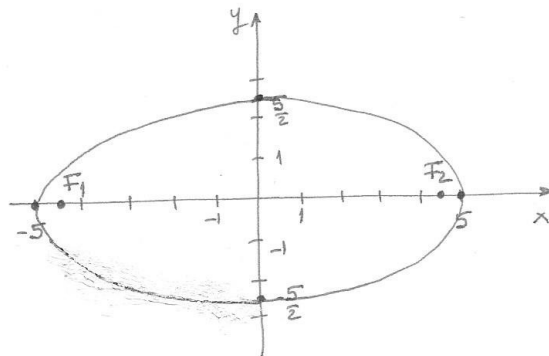
$e^2 = 25 - \dfrac{25}{4}$

$e^2 = \dfrac{100 - 25}{4}$

$e^2 = \dfrac{75}{4} \;/\sqrt{}$

$\boxed{e} = \dfrac{\sqrt{75}}{2} = \dfrac{\sqrt{25\cdot 3}}{2} = \boxed{\dfrac{5\sqrt{3}}{2}}$

tj.: $\boxed{\begin{array}{l} A(-5,0),\, B(5,0) \\ C\left(,-\dfrac{5}{2}\right),\, D\left(0,\dfrac{5}{2}\right) \end{array}}$

žar.: $\boxed{F_1\left(-\dfrac{5\sqrt{3}}{2},0\right),\, F_2\left(\dfrac{5\sqrt{3}}{2},0\right)}$

2) Napiši jednadžbu elipse koja prolazi točkama $A\left(\underset{x}{4},\underset{y}{\tfrac{3}{2}}\right)$ i $B\left(\underset{x}{3},\underset{y}{2}\right)$.

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\dfrac{16}{a^2} + \dfrac{\frac{9}{4}}{b^2} = 1$

$\dfrac{9}{a^2} + \dfrac{4}{b^2} = 1$

$16 \cdot \dfrac{1}{a^2} + \dfrac{9}{4} \cdot \dfrac{1}{b^2} = 1$

$9 \cdot \dfrac{1}{a^2} + 4 \cdot \dfrac{1}{b^2} = 1$

$\dfrac{1}{a^2} = u, \; \dfrac{1}{b^2} = v$

$16u + \dfrac{9}{4}v = 1 \;/\cdot 4$

$9u + 4v = 1$

$64u + 9v = 4 \;/\cdot 4$

$9u + 4v = 1 \;/\cdot(-9)$

$256u + 36v = 16$
$-81u - 36v = -9$ $\Big|+$

$175u = 7 \;/:175$

$u = \dfrac{7}{175}^{1}{}_{25}$

$u = \dfrac{1}{25}$

$9 \cdot \dfrac{1}{25} + 4v = 1$

$\dfrac{9}{25} + 4v = 1$

$4v = 1 - \dfrac{9}{25}$

$4v = \dfrac{25-9}{25}$

$4v = \dfrac{16}{25} \;/:4$

$v = \dfrac{4}{25}$

$\dfrac{1}{a^2} = \dfrac{1}{25} \Rightarrow a^2 = 25$

$\dfrac{1}{b^2} = \dfrac{4}{25} \Rightarrow b^2 = \dfrac{25}{4}$

$\boxed{\dfrac{x^2}{25} + \dfrac{y^2}{\frac{25}{4}} = 1}$

3) Napiši jednadžbu elipse u kojoj je $a=5$, $|F_1 F_2| = 8$.

$|F_1 F_2| = 2e = 8 \Rightarrow e = 4$

$a = 5$

$b^2 = a^2 - e^2 = 25 - 16 = 9$

$b^2 = 9 \ /\sqrt{}$

$\boxed{b = 3}$

$$\boxed{\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1}$$

4) Napiši jednadžbu elipse kojoj je jedno tjeme $A(-5,0)$, a prolazi točkom $(4, \frac{6}{5})$.
$\hookrightarrow a = 5$

$\quad x \quad y$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\dfrac{16}{a^2} + \dfrac{\frac{36}{25}}{b^2} = 1$

$\dfrac{16}{25} + \dfrac{36}{25b^2} = 1 \quad / \cdot 25b^2$

$16b^2 + 36 = 25b^2$

$\quad -9b^2 = -36 \ /: (-9)$

$\boxed{b^2 = 4}$

$$\boxed{\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1}$$

5) Napiši jednadžbu elipse kojoj je $a+b=16$ i linearni ekscentricitet $e = 8$.

$a + b = 16 \Rightarrow b = 16 - a$

$\underline{e = 8}$

$a^2 = b^2 + e^2$

$a^2 = (16 - a)^2 + 8^2$

$a^2 = 256 - 32a + a^2 + 64$

$32a = 256 + 64$

$32a = 320 \ /: 32$

$\boxed{a = 10}$

$b = 16 - a$

$\boxed{b = 6}$

$$\boxed{\dfrac{x^2}{100} + \dfrac{y^2}{36} = 1}$$

6) Odredi međusobni položaj pravca $y = \frac{1}{2}x + \frac{5}{2}$ i elipse $4x^2 + 9y^2 = 36$.

$4x^2 + 9y^2 = 36$ ⤶

$y = \frac{1}{2}x + \frac{5}{2}$ ⟵

$4x^2 + 9\left(\frac{1}{2}x + \frac{5}{2}\right)^2 = 36$

$4x^2 + 9\left(\frac{1}{4}x^2 + 2 \cdot \frac{1}{2}x \cdot \frac{5}{2} + \frac{25}{4}\right) = 36$

$4x^2 + \frac{9}{4}x^2 + \frac{45}{2}x + \frac{225}{4} = 36 \quad / \cdot 4$

$16x^2 + 9x^2 + 90x + 225 = 144$

$25x^2 + 90x + 81 = 0$

$D = b^2 - 4ac$

$D = 90^2 - 4 \cdot 25 \cdot 81$

$D = 8100 - 8100$

$D = 0$

$\Rightarrow$ pravac i elipsa imaju jednu zajedničku točku

$\Rightarrow$ $\boxed{\text{pravac je tangenta elipse}}$

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7) Odredi presječne točke pravca $x = 6$ i elipse $x^2 + 4y^2 = 100$.

$x^2 + 4y^2 = 100$ ⤶

$x = 6$ ⟵

$36 + 4y^2 = 100$

$4y^2 = 100 - 36$

$4y^2 = 64 \quad /: 4$

$y^2 = 16 \quad / \sqrt{}$

$y_{1,2} = \pm 4$

$\boxed{S_1(6, 4)}$

$\boxed{S_2(6, -4)}$

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8) Odredi jednadžbu tangente i normale elipse $4x^2 + 25y^2 = 100 \quad /: 100$ u njenoj točki $D\left(x < 0, \frac{6}{5}\right)$.

$\frac{x^2}{25} + \frac{y^2}{4} = 1$

$4x^2 + 25y^2 = 100$

$4x^2 + 25 \cdot \frac{36}{25} = 100$

$4x^2 + 36 = 100$

$4x^2 = 100 - 36$

$4x^2 = 64 \quad /: 4$

$x^2 = 16 \quad / \sqrt{}$

$x_1 = 4, \quad \boxed{x_2 = -4}$

$D\left(-4, \frac{6}{5}\right)$

$\frac{x_0 \cdot x}{a^2} + \frac{y_0 \cdot y}{b^2} = 1$

$-\frac{4x}{25} + \frac{\frac{6}{5}y}{4} = 1$

$-\frac{4x}{25} + \frac{6y}{20} = 1$

$-\frac{4x}{25} + \frac{3y}{10} = 1 \quad / \cdot 50$

$-8x + 15y = 50$

$15y = 8x + 50 \quad /: 15$

$y = \frac{8}{15}x + \frac{50}{15}$

$t \ldots \boxed{y = \frac{8}{15}x + \frac{10}{3}}$

$k_t = \frac{8}{15}$

$k_n = -\frac{15}{8}, \quad D\left(-4, \frac{6}{5}\right)$

$y - y_1 = k(x - x_1)$

$y - \frac{6}{5} = -\frac{15}{8}(x + 4)$

$y - \frac{6}{5} = -\frac{15x}{8} - \frac{15}{2}$

$y = -\frac{15}{8}x - \frac{15}{2} + \frac{6}{5}$

$y = -\frac{15}{8}x + \frac{-75 + 12}{10}$

$n \ldots \boxed{y = -\frac{15}{8}x - \frac{63}{10}}$

9) Kako glase jednadžbe tangenti elipse $4x^2 + 9y^2 = 36$, koje su okomite na pravac $y = -\frac{9}{8}x + 5$?

$4x^2 + 9y^2 = 36 \ /:36$

$\frac{1}{9}\frac{4x^2}{36} + \frac{9y^2}{36}_4 = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$a^2 = 9, \quad b^2 = 4$

$k = \frac{8}{9}$

$a^2k^2 + b^2 = l^2$

$\overset{1}{\cancel{9}} \cdot \frac{64}{\cancel{81}_9} + 4 = l^2$

$\frac{64}{9} + 4 = l^2$

$\frac{64 + 36}{9} = l^2$

$\frac{100}{9} = l^2$

$l^2 = \frac{100}{9} \ /\sqrt{}$

$l_{1,2} = \pm \frac{10}{3}$

$t_1 \ldots \boxed{y = \frac{8}{9}x + \frac{10}{3}}$

$t_2 \ldots \boxed{y = \frac{8}{9}x - \frac{10}{3}}$

10) Pod kojim se kutom iz točke $T(7, -2)$ vidi elipsa $\frac{x^2}{16} + \frac{y^2}{9} = 1$?

$a^2 = 16, \quad b^2 = 9$

$y = kx + l$

$T(7, -2)$

$-2 = 7k + l$

$-l = 7k + 2 \ /:(-1)$

$l = -7k - 2$

$a^2k^2 + b^2 = l^2 \quad\longrightarrow\quad = (7k+2)^2$

$16k^2 + 9 = (-7k-2)^2$

$16k^2 + 9 = 49k^2 + 28k + 4$

$16k^2 + 9 - 49k^2 - 28k - 4 = 0$

$-33k^2 - 28k + 5 = 0 \ /:(-1)$

$33k^2 + 28k - 5 = 0$

$k_{1,2} = \frac{-28 \pm \sqrt{28^2 - 4 \cdot 33 \cdot (-5)}}{2 \cdot 33} = \frac{-28 \pm \sqrt{784 + 660}}{66}$

$= \frac{-28 \pm \sqrt{1444}}{66} = \frac{-28 \pm 38}{66}$

$k_1 = \frac{-28 + 38}{66} = \frac{10}{66} = \frac{5}{33}$

$k_2 = \frac{-28 - 38}{66} = -\frac{66}{66} = -1$

$tg\,\varphi = \left| \frac{k_2 - k_1}{1 + k_1 \cdot k_2} \right| = \left| \frac{-1 - \frac{5}{33}}{1 + \frac{5}{33} \cdot (-1)} \right| = \left| \frac{\frac{-33 - 5}{33}}{1 - \frac{5}{33}} \right| = \left| \frac{\frac{-38}{33}}{\frac{33 - 5}{33}} \right| =$

$= \left| \frac{\frac{-38}{\cancel{33}}^{19}}{\frac{\cancel{28}}{\cancel{33}}_{14}} \right| = \left| -\frac{19}{14} \right| = \frac{19}{14}$

$\varphi = tg^{-1}\left(\frac{19}{14} \right)$

$\boxed{\varphi = 53° 36' 56''}$